1. Mathematical Logic

- **Mathematically Acceptable Statement**

A sentence is called a mathematically acceptable statement if it is either true or false, but not both.

We denote statements by small letters p, q, r,s, etc.

Example: Consider the following sentences:

- "Complete your homework."

It is not a statement as it is an order.

- "I can't wait to open my present!"

It is not a statement as it is an exclamatory sentence.

- "The value of iota is –1."

It is a statement. Since we know that $i = \sqrt{-1}$, the given sentence is false.

- "He is going to America."

It is not a statement as in this sentence we do not know who "he" is.

- "The number 0 is the smallest whole number."

It is a statement as this sentence is always true.

- "Do you have internet at your home?"

It is not a statement as it is a question.

- "The probability of getting a head when a coin is tossed is $\frac{1}{3}$"

It is a statement as this sentence is always false. The probability of getting a head when a coin is tossed is $\frac{1}{2}$.

Note:

- Sentences involving variable time such as today, tomorrow and yesterday are not statements.
- Sentences involving an exclamation, a question or an order are not statements.
- Sentences involving pronouns such as he or she, unless a particular person is referred to, are not statements.
- Sentences involving pronouns for variable places such as here and there are not statements.

- **Compound Statement**

A compound statement is one that is made up of two or more statements. In this case, each smaller statement is called the component of the compound statement. These component statements are joined by words such as "And" and "Or". These are called connectors or connecting words.

Example: The statement "27 is a multiple of 9 and it is even" is a compound statement.
Its component statements are:
p: "27 is a multiple of 9."
q: "27 is even."
Here, the connecting word is "And".

- **Connectives in Compound Statements**

A statement with "And" is not always a compound statement.

Example: Water can be prepared by the mixture of hydrogen and oxygen in a certain ratio. This statement is not a compound statement.

- **Rules regarding the connector "And"**
  - The compound statement with the connector "And" is true if all its component statements are true.
  - The compound statement with the connector "And" is false if any/both of its component statements is/are false.

  Example: The compound statement "27 is a multiple of 9 and it is even" is false. The component statement "27 is a multiple of 9" is true; however, the other component statement "27 is even" is false.

- **Rules regarding the connector "Or"**
  - A compound statement with the connector "Or" is true when one component statement is true, or both the component statements are true.
  - A compound statement with the connector "Or" is false when both the component statements are false.

  Example: The compound statement "The equation $x^2 - 1 = 0$ is true for $x = 1$ or $x = -1$" is true. This is because both its component statements "The equation $x^2 - 1 = 0$ is true for $x = 1$" and "The equation $x^2 - 1 = 0$ is true for $x = -1$" are true.

- **Types of "Or"**
  - Exclusive "Or": A compound statement with the connector "Or" in which either of the component statements may be true, but not both

  Example: A student can take home science or painting as his/her additional subject in class XI.

  - Inclusive "Or": A compound statement with the connector "Or" in which either of the component statements or both may be true.

  Example: In an equilateral triangle, all the three sides are of equal length or all the three angles are of equal measure.

- **Negations of Statements**
  - The denial of a statement is called the negation of the statement.

- If p is a statement, then the negation of p is also a statement, and is denoted by $\sim p$, and is read as "not p".
- While writing the negation of a statement, phrases such as "It is not the case" or "It is false that" are used.

Example 1: Write the negation of the following statement:
p: The square root of every positive number is positive.
Solution:
The negation of the given statement can be written as:
The square root of every positive number is not positive.
Or
It is false that the square root of every positive number is positive.
Or
It is not the case that the square root of every positive number is positive.
Or
There exists a positive number whose square root is not positive.

- **Validity of "If-then" and "If and only if" statements**
  - In order to show that the statement "p and q" is true, the following steps are followed:

    Step 1: Show that statement p is true.

    Step 2: Show that statement q is true.

  - In order to show that the statement "p or q" is true, the following cases are to be considered:

    Case 1: Assuming that p is false, show that q must be true

    Case 2: Assuming that q is false, show that p must be true

  - In order to prove the statement "if p, then q', we need to show that any one of the following cases is true.

    Case 1: Assuming that p is true, prove that q must be true. (Direct method)

    Case 2: Assuming that q is false, prove that p must be false. (Contrapositive method)

Example 1: Prove the following statement: "If r is irrational, then $\sqrt[3]{r}$ is irrational."
Solution: We shall prove the contrapositive of the given statement.

This means that we will prove if $\sqrt[3]{r}$ is rational, then r is rational.
Let $\sqrt[3]{r}$ be rational.
$\Rightarrow \sqrt[3]{r} = \frac{a}{b}$, for some integers a and b, and $b \neq 0$
$\Rightarrow r = \frac{a^3}{b^3}$
Now, since a and b are integers, a^3 and b^3 are also integers. Since $b \neq 0$, $b^3 \neq 0$.

Hence, r is rational.

Thus, if r is irrational, then $\sqrt[3]{r}$ is irrational.

Statement Pattern

A statement formed using simple statements, such as p, q and r, and one or more connectives, such as $\wedge, \vee, \sim, \rightarrow$ and $\leftrightarrow$ ∧,∨,~,→ and↔, is called a statement pattern.

Examples: $p \wedge \wedge \sim\sim q$, $p \vee (p \wedge q)$ ∨p∧q and $p \leftrightarrow q$p↔q

Logical Equivalence

Two statement patterns, say, S_1 and S_2, are said to be logically equivalent if they have the same truth values in their respective columns in the joint truth table. If S_1 and S_2 are equivalent, then they are written either as $S_1 \equiv \equiv S_2$ or as $S_1 = S_2$.

Compound statements (or propositions) that are true (T) for any truth value of their components are called **tautologies**.

The negation of a tautology is called a **fallacy** or a **contradiction**—that is, a proposition that is false (F) for any truth value of its components is called a fallacy.

A compound statement (or proposition) that is neither a tautology nor a contradiction is called a **contingency**.

- **Quantifiers in Compound Statements**

  Some statements may contain special phrases such as "There exists", "For all", "For every". These are called quantifiers.

  - The quantifier for the statement "For every prime number x, $n \in \mathrm{N}$, x^n has exactly $(n + 1)$ factors" is "For every". This statement is equivalent to "If A is the set of all prime numbers, then for any number x in the set A and $n \in \mathrm{N}$, x^n has exactly $(n + 1)$ factors".
  - The quantifier for the statement "There exists a number which is divisible by 4 and 7, but not by 2" is "There exists". This statement is equivalent to "Out of all numbers that are divisible by both 4 and 7, there is at least one number which is not divisible by 2".

Duality

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge\wedge$ by $\vee\vee$, t by c and c by t, where t denotes a tautology and c denotes a contradiction.

Principle of Duality

If a compound statement S_1 contains only $\sim\sim$, $\wedge\wedge$ and $\vee\vee$ and statement S_2 arises from S_1 by replacing $\wedge\wedge$ by $\vee\vee$ and $\vee\vee$ by $\wedge\wedge$, then S_1 is a tautology iff S_2 is a contradiction.**Negation of Compound Statements**

The negation of a statement is a statement that changes the truth value of the given statement. A negation only negates the statement that immediately follows it. Now, let us learn negations of compound statements involving conjunction, disjunction, conditional, biconditional, etc.

i) **Negating a Conjunction** (and)

The negation of the conjunction of two simple statements is the disjunction of their negations.

$\sim(p \wedge q) \equiv \sim p \vee \sim q$ ~p ∧ q ≡ ~p ∨~q

ii) **Negating a Disjunction** (or)

The negation of the disjunction of two simple statements is the conjunction of their negations.

$\sim(p \vee q) \equiv \sim p \wedge \sim q$ ~p ∨ q ≡ ~p ∧~q

iii) **Negating a Negation**

Let *p* be a simple statement. Then, the negation of the negation of simple statement *p* is *p* itself.

$\sim(\sim p) \equiv p$ ~~p ≡p

iv) **Negating a Conditional** (if ...then)

The negation of a conditional statement is given by $\sim(p \rightarrow q) \equiv (p \wedge \sim q)$ ~p →q ≡ p∧~q.

v) **Negating a Biconditional** (if and only if)

The negation of a biconditional statement is given by
$\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ ~p ↔q ≡ p∧~q ∨ q∧~p.

vi) **Negating a Quantified Statement**

Negate an "all" statement by changing the quantifier to "some" and negating the rest of the statement.
Negate a "some" statement by changing the quantifier to "all" and negating the rest of the statement or changing "some" to "no".

Also, while doing the negation of quantified statements, we replace "for every" by "there exist" and vice versa.

Note:

i) Negation of a statement involving one or more simple statements such as *p, q* and *r* and one or more connectives such as $\wedge, \vee$ and $\sim$ ∧, ∨ and ~ can be obtained by replacing ∧∧ by ∨∨, ∨∨ by ∧∧ and *p, q, r,...* by $\sim p, \sim q, \sim r, \ldots$ ~p, ~q, ~r, ...
ii) $p \wedge q \wedge r$ p∧q∧r is true if and only if *p, q* and *r* are all true and $p \wedge q \wedge r$ p∧q∧r is false even if one of them is false.
iii) $p \vee q \vee r$ p∨q∨r is false if and only if *p, q* and *r* are all false and $p \vee q \vee r$ p∨q∨r is true even if one of them is true.**Application of Logic to Switching Circuits**

In an electric circuit, switches are connected by wires. A signal can go from left to right through a switch when it is closed. No signal can pass when the switch is open.

An electric switch is a two-state device used for turning current ON or OFF.

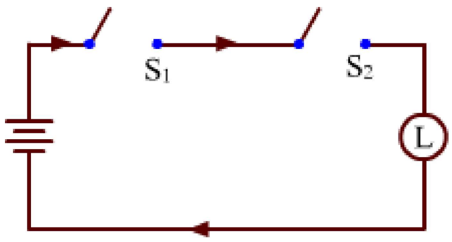
On (closed) switch

Off (open) switch

If the switch is ON, its truth value is 1; and if the switch is OFF, its truth value is 0.

Switches can be put together by connecting them in series or in parallel.

i) **Switches in Series**

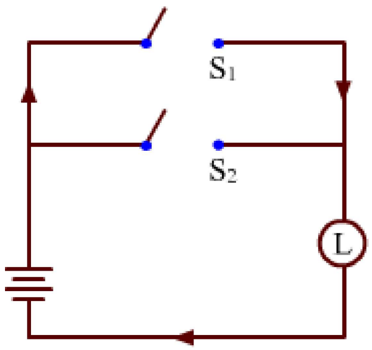


Two switches in series

$S_1 \wedge S_2$

The series connection corresponds to S_1 and S_2. This can be written in the logic notation as $S_1 \ \wedge \ S_2$ $S1 \wedge S2$.

ii) **Switches in Parallel**



Two switches in parallel

$S_1 \vee S_2$

The parallel connection corresponds to S_1 or S_2. This can be written in the logic notation as $S_1 \ \vee \ S_2$ $S1 \vee S2$.

Note:

i) It is possible that there may be two or more switches in a circuit that open or close simultaneously. Such switches are denoted by the same letter and are called equivalent switches.

ii) Any switch S in a circuit can be coupled with switch S' having opposite effect to that of S. If S is open, then S' is closed and vice versa. If p is the statement letter that corresponds to switch S, then $\sim\sim p$ is the statement letter that corresponds to switch S'. The switches S and S' are called complementary switches.

iii) Two circuits are said to be equivalent if the output of two circuits is the same in all possible ways. Also, one circuit is called simpler if it contains fewer number of switches.

iv) Negation state: The switches denoted by S and S' are said to be related by their states such that if S is ON (closed), then S' is OFF (open) and vice versa. Switch S' (or $\sim\sim$S) is called the negation of S.